4th Benelux Mathematical Olympiad

20–22 April 2012 — Namur, Belgium

Solutions

Problem 1. A sequence $a_1, a_2, \ldots, a_n, \ldots$ of natural numbers is defined by the rule

$$a_{n+1} = a_n + b_n \quad (n = 1, 2, \ldots)$$

where b_n is the last digit of a_n. Prove that such a sequence contains infinitely many powers of 2 if and only if a_1 is not divisible by 5.

Solution. First we can observe that:

- If a_1 is divisible by 5, then $a_n = a_2 = 0 \pmod{10}$ $\forall n \geq 2$.
- If a_1 is not divisible by 5, then for $n \geq 2$: a_n is even, the sequence b_n is periodic, its period is a cyclic permutation of $(2, 4, 8, 6)$, and $a_{n+4} = a_n + 20$.

(a) Let us suppose that a_1 is divisible by 5.
Since $2^k \neq 0 \pmod{10}$ for any $k \in \mathbb{N}$, the sequence does not contain any power of 2 for $n \geq 2$.

(b) Let us suppose that a_1 is not divisible by 5.
We can remark that the sequence of powers of 2 modulo 20 respects the period $(12, 4, 8, 16)$ starting with $2^5 = 32$. We choose j such that $a_j = 2 \pmod{10}$ (i.e. $b_j = 2$) and look at the parity of its penultimate digit.

- If $a_j = 12 \pmod{20}$, then the numbers a_{j+4k}, $k \in \mathbb{N}$, represent all the numbers congruent to 12 $\pmod{20}$ and greater than a_j, so all powers of 2 congruent to 12 $\pmod{20}$ and greater than a_j appear in the sequence.
- If $a_j = 2 \pmod{20}$, then the numbers a_{j+1+4k}, $k \in \mathbb{N}$, represent all the numbers congruent to 4 $\pmod{20}$ and greater than a_{j+1}, so all powers of 2 congruent to 4 $\pmod{20}$ and greater than a_{j+1} appear in the sequence.

Thus, the sequence contains infinitely many powers of 2.

Alternative 1 for (b). We choose j such that $a_j = 2 \pmod{10}$ (i.e. $b_j = 2$).

- If $a_j = 20t + 12$ for some $t \in \mathbb{N}$, then $a_{j+4k} = a_j + 20k = 20(t + k) + 12$, $\forall k \in \mathbb{N}$. We obtain infinitely many powers of 2 by taking $k = \frac{2^{4s+3}-3}{5} - t$ (with $s \in \mathbb{N}$ large enough to have $k > 0$) since $2^{4s+3} = 3 \pmod{5}$, $\forall s \in \mathbb{N}$.

- If $a_j = 20t + 2$ for some $t \in \mathbb{N}$, then $a_{j+1+4k} = a_{j+1} + 20k = 20(t+k) + 4$, $\forall k \in \mathbb{N}$. We obtain infinitely many powers of 2 by taking $k = \frac{2^{4s}-1}{5} - t$ (with $s \in \mathbb{N}$ large enough to have $k > 0$) since $2^{4s} = 1 \pmod 5$, $\forall s \in \mathbb{N}$.

Alternative 2 for (b). Choose j such that a_j is a multiple of 4, i.e. $a_j = 4q$ (such a j always exists since $a_{n+1} = a_n + 2$ for infinitely many n). Then we have $a_{j+4k} = a_j + 20k = 4(q+5k)$. Let us look for (k, m) such that

$$a_{j+4k} = 2^m \iff 4(q+5k) = 2^m \iff q + 5k = 2^{m-2} \iff 2^{m-2} = q \pmod 5.$$

Since q could not be a multiple of 5, we have $q \in \{1, 2, 3, 4\} \pmod 5$. Since the sequence $2^{m-2} \pmod 5$ is periodic with period $(1, 2, 4, 3)$, we find that $2^{m-2} = q \pmod 5$ happens for infinitely many values of m. Hence $2^{m-2} = q + 5k$ is solvable for infinitely many pairs (k, m). Noting that m determines k and that k is nonnegative as soon as m is large enough concludes the proof.

Alternative 3 for (b). We shall show that for any $n > 1$ there is some $k \geq n$ such that a_k is a power of 2. First, we observe that we can always find $m \in \{n, n+1, n+2, n+3\}$ such that a_m is divisible by 4. If a_m is not a power of 2, we write $a_m = 2^b c$ with $b \geq 2$ and $c > 1$ odd. Then we have

$$a_{m+4\cdot 2^{b-2}} = a_m + 20\left(2^{b-2}\right) = 2^b c + 5 \cdot 2^b = 2^{b+1} \frac{c+5}{2}.$$

If $c > 5$, we have $\frac{c+5}{2} < c$ and hence the odd factor of $a_{m+4.2^{b-2}}$ is strictly smaller than the odd factor of a_m. Therefore there is some $m' > m$ such that $a_{m'} = 2^{b'} c'$ with c' odd and ≤ 5. The case $c' = 5$ is forbidden. If $c' = 1$, then $a_{m'}$ is a power of 2. If $c' = 3$, then $a_{m'+4.2^{b'-2}} = 2^{b'+3}$ is a power of 2.

Problem 2. Find all quadruples (a, b, c, d) of positive real numbers such that $abcd = 1$, $a^{2012} + 2012b = 2012c + d^{2012}$ and $2012a + b^{2012} = c^{2012} + 2012d$.

Solution. Rewrite the last two equations into

$$a^{2012} - d^{2012} = 2012(c - b) \text{ and } c^{2012} - b^{2012} = 2012(a - d) \tag{1}$$

and observe that $a = d$ holds if and only if $c = b$ holds. In that case, the last two equations are satisfied, and condition $abcd = 1$ leads to a set of valid quadruples of the form $(a, b, c, d) = (t, \frac{1}{t}, \frac{1}{t}, t)$ for any $t > 0$.

We show that there are no other solutions. Assume that $a \neq d$ and $c \neq b$. Multiply both sides of (1) to obtain

$$(a^{2012} - d^{2012})(c^{2012} - b^{2012}) = 2012^2(c - b)(a - d)$$

and divide the left-hand side by the (nonzero) right-hand side to get

$$\frac{a^{2011} + \cdots + a^{2011-i}d^i + \cdots + d^{2011}}{2012} \cdot \frac{c^{2011} + \cdots + c^{2011-i}b^i + \cdots + b^{2011}}{2012} = 1 \ .$$

Now apply the arithmetic-geometric mean inequality to the first factor

$$\frac{a^{2011} + \cdots + a^{2011-i}d^i + \cdots + d^{2011}}{2012} > \sqrt[2012]{(ad)^{\frac{2011\times 2012}{2}}} = (ad)^{\frac{2011}{2}} \ .$$

The inequality is strict, since equality holds only if all terms in the mean are equal to each other, which happens only if $a = d$. Similarly, we find

$$\frac{c^{2011} + \cdots + c^{2011-i}b^i + \cdots b^{2011}}{2012} > \sqrt[2012]{(cb)^{\frac{2011\times 2012}{2}}} = (cb)^{\frac{2011}{2}} \ .$$

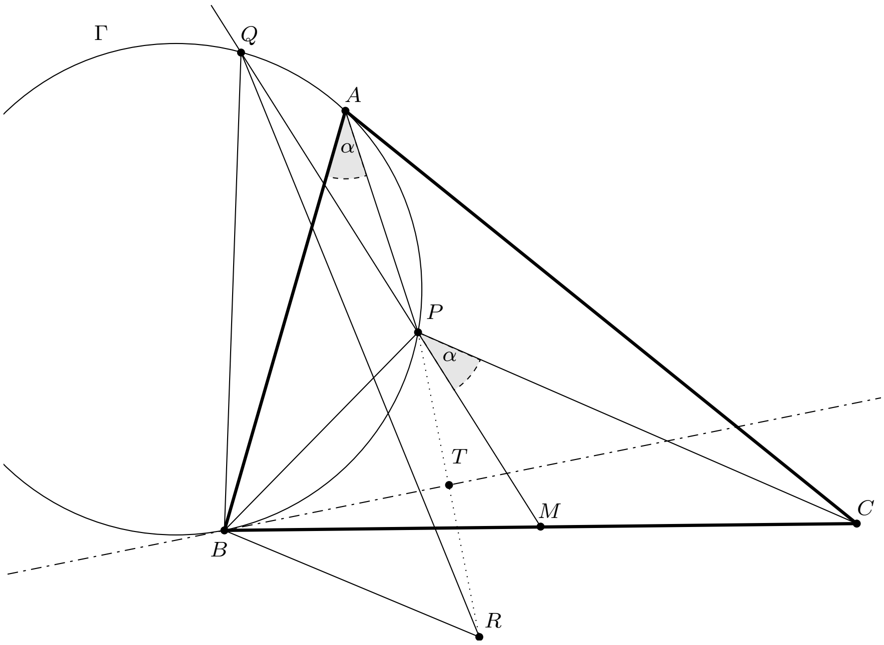
Multiplying both inequalities, we obtain

$$(ad)^{\frac{2011}{2}}(cb)^{\frac{2011}{2}} < 1$$

which is equivalent to $abcd < 1$, a contradiction.

Problem 3. In triangle ABC the midpoint of BC is called M. Let P be a variable interior point of the triangle such that $\angle CPM = \angle PAB$. Let Γ be the circumcircle of triangle ABP. The line MP intersects Γ a second time in Q. Define R as the reflection of P in the tangent to Γ in B. Prove that the length $|QR|$ is independent of the position of P inside the triangle.

Solution. We claim $|QR| = |BC|$, which will clearly imply that quantity $|QR|$ is independent from the position of P inside triangle $\triangle ABC$ (and independent from the position of A).
This equality will follow from the equality between triangles $\triangle BPC$ and $\triangle RBQ$. This in turn will be shown by means of three equalities (two sides and an angle): $|BP| = |RB|$, $|PC| = |BQ|$ and $\angle BPC = \angle RBQ$.



(a) $|BP| = |RB|$

Obvious since R is the reflection of P in a line going through B.

(b) $|PC| = |BQ|$

Let U be the fourth vertex of parallelogram $BPCU$. Then U is on line PQ and $\angle BUP = \angle UPC = \alpha$. If Q is on the same arc PB as A, then $\angle BQP = \alpha$, and $\triangle QPU$ is isosceles; hence, $|BQ| = |BU| = |PC|$. On the other way, if Q is on the other arc PB, then $\angle BQP$ and α are supplementary, hence $BQU = \alpha$, and again $\triangle QPU$ is isosceles; the same conclusion follows.

(c) $\angle BPC = \angle RBQ$

Define T to be the midpoint of PR. Then line BT, tangent to circle Γ in B, splits $\angle RBQ$ into two parts, $\angle RBT$ and $\angle TBQ$.

We first show that $\angle RBT = \alpha$. Indeed, by symmetry, $\angle RBT = \angle PBT$ and, since BT is tangent to Γ, we have that $\angle PBT = \angle PAB$ (because they both intercept the same arc $\widehat{PB}$ on circle Γ), from which our claim follows.

We then show that $\angle TBQ = \angle BPM$. Indeed, since $\angle TBQ$ and $\angle BPQ$ intercept opposite arcs on circle Γ, they are supplementary and we have $\angle TBQ = \pi - \angle BPQ = \angle BPM$.

We finally conclude that

$$\angle RBQ = \angle RBT + \angle TBQ = \alpha + \angle BPM = \angle MPC + \angle BPM = \angle BPC.$$

We have thus shown $\triangle BPC = \triangle RBQ$, which completes the proof.

Note. Notice that point A does not play any role in the problem except fixing circle Γ (and, for that reason, the result is also valid when P is chosen outside of triangle $\triangle ABC$).

Alternative 1 for (b). The law of sines in triangle $\triangle BQM$ gives

$$\frac{|BM|}{\sin \angle BQM} = \frac{|BQ|}{\sin \angle BMQ} \; . \tag{2}$$

Since Q belongs to circle Γ, we have either $\angle BQP = \angle BAP = \alpha$, hence $\angle BQM = \angle MPC$, or these angles are supplementary; in both cases they have equal sines. We also have that $\angle BMQ$ and $\angle CMP$ are supplementary, hence have equal sines. Using these facts along with $|BM| = |MC|$ transforms (2) into

$$\frac{|MC|}{\sin \angle MPC} = \frac{|BQ|}{\sin \angle CMP}$$

from which the law of sines in triangle $\triangle CPM$ implies that $|BQ| = |PC|$.

Alternative 2 for (b). Let S be the second intersection of line CP with circle Γ. Then, $\angle BSP = \alpha$, so BS and MP are parallel; since M is the midpoint of segment BC, P is the midpoint of SC. If Q is on the same arc PB as A, then the quadrilateral $QPBS$ is an isosceles trapezoid, and $|QB| = |SP| = |PC|$. If Q is on the other arc PB, then the quadrilateral $PQBS$ is an isosceles trapezoid, and again $|QB| = |SP| = |PC|$.

Problem 4. Yesterday, $n \geq 4$ people sat around a round table. Each participant remembers only who his two neighbours were, but not which one sat on his left and which one sat on his right. Today, you would like the same people to sit around the same round table so that each participant has the same two neighbours as yesterday (it is possible that yesterday's left-hand side neighbour is today's right-hand side neighbour). You are allowed to query some of the participants: if anyone is asked, he will answer by pointing at his two neighbours from yesterday.

(a) Determine the minimal number $f(n)$ of participants you have to query in order to be certain to succeed, if later questions must not depend on the outcome of the previous questions. That is, you have to choose in advance the list of people you are going to query, before effectively asking any question.

(b) Determine the minimal number $g(n)$ of participants you have to query in order to be certain to succeed, if later questions may depend on the outcome of previous questions. That is, you can wait until you get the first answer to choose whom to ask the second question, and so on.

Solution.

(a) $f(n) = n - 3$.

- Asking $n-4$ questions is not enough since the $n-4$ people queried might be sitting in a consecutive string, in which case the $n-4$ answers allow one to sit $n-2$ people in the same positions as yesterday, but there is still an ambiguity among the two remaining ones.
- Let us show that $n-3$ questions suffice. Among the 3 people who are not queried, at least 2 must sit next to people who have been queried. If exactly 2 do, then both these people must be neighbours of the third, so that the neighbours of everybody are known and we are done. If all 3 unqueried people sit next to a queried person, then at least one of them has two queried neighbours, and again it follows that the neighbours of everybody are known, so that we are done.

(b) $g(n) = n - 1 - \left\lceil \frac{n}{3} \right\rceil$ ($= n - 1 - \left\lfloor \frac{n+2}{3} \right\rfloor = \left\lfloor \frac{2n}{3} \right\rfloor - 1 = \left\lceil \frac{2n-5}{3} \right\rceil$).

Say there is a *link* between two people if and only if they are neighbours. There are in total n links, which we all need to identify. By asking a person for his neighbours, we can discover at most two new links. More precisely, if at any point we query a participant who has not yet been pointed as a neighbour, we discover exactly two new links (we call this a type-0 query). If we query a participant who has been pointed once as a neighbour, will discover exactly one new link (we call this a type-1 query). Of course, querying a participant who has already been pointed twice provides no information (and we assume in the rest of this solution that it never happens).

First note that, since $f(4) = 1$, we also have $g(4) = 1$. We now prove the formula for $g(n)$ for $n \geq 5$.

- Let us show that $n-1-\left\lceil\frac{n}{3}\right\rceil$ questions suffice. Our strategy consists in making sure that the first $\left\lceil\frac{n}{3}\right\rceil$ queries are type-0. Let us show that this is always possible. A type-0 query requires a participant that hasn't been queried or pointed before. Since the number of those participants decreases by three at most after each query, we see that it is always possible to perform $\left\lceil\frac{n}{3}\right\rceil$ type-0 queries first. During this phase we discover $2\left\lceil\frac{n}{3}\right\rceil$ links.

  The remaining queries will be either type-0 or type-1, and each of them discovers at least one new link. We perform them until $n-1$ links have been discovered, after which we are done (the last link can be deduced without query). The number of queries in this second phase is therefore at most[1] $n-1-2\left\lceil\frac{n}{3}\right\rceil$, and the total is at most $\left\lceil\frac{n}{3}\right\rceil+(n-1-2\left\lceil\frac{n}{3}\right\rceil)=n-1-\left\lceil\frac{n}{3}\right\rceil$.

- We now show that $n-2-\left\lceil\frac{n}{3}\right\rceil=\hat{g}(n)$ questions are not enough.

  (i) Consider the pool of unqueried and unpointed participants ; each type-0 must query this pool. Since, from the point of view of the questioner, all elements of the pool are undistinguishable, we can assume that each type-0 query asks the second leftmost participant in the pool (except if there is only one element left in the pool). One can then check that the pool, which starts as a string of n contiguous participants, will stay contiguous after each type-0 and type-1 query. Furthermore, using our assumption, we see that each type-0 query removes three participants from the pool. Therefore there can be at most $\left\lceil\frac{n}{3}\right\rceil$ type-0 queries in the scenarios corresponding to our assumption.

  (ii) Assume there are k type-0 queries. Since there are $\hat{g}(n)$ queries, the number of discovered links is equal to $2k+(\hat{g}(n)-k)=\hat{g}(n)+k=n-2+k-\left\lceil\frac{n}{3}\right\rceil$. If k is strictly less than $\left\lceil\frac{n}{3}\right\rceil$, we discover strictly less than $n-2$ links, which is clearly insufficient (indeed, there are at least three missing links, and one can check that whatever the configuration of the missing links, there are always several orders compatible with the discovered links).

  (iii) We now analyze the remaining case with $k=\left\lceil\frac{n}{3}\right\rceil$ type-0 queries[2], in which we discover $n-2$ links. On the one hand, if the missing links are disjoint, there are always two orders compatible with the discovered links (for example when $n=7$ and links are missing between the $(4,5)$ and $(7,1)$ pairs of neighbours, the two orders are $1-2-3-4\quad 5-6-7$ and $1-2-3-4\quad 7-6-5$). On the other hand, a situation where the two missing links would be adjacent would allow the identification of the correct order. However, this never happens in the scenarios corresponding to the assumption we made in (i). Indeed, two adjacent missing links imply that some participant is unqueried and unpointed at the end of the process. Since we perform $k=\left\lceil\frac{n}{3}\right\rceil$ type-0 queries (the maximum), the reasoning from (i) shows that the pool of unqueried and unpointed participants is empty at the end of the process, which contradicts the existence of two adjacent missing links.

[1] Here we use the assumption $n\geq 5$, since quantity $n-1-2\left\lceil\frac{n}{3}\right\rceil$ is negative when $n=4$.
[2] Note that this cannot happen when $n\in\{4,5,7\}$ since we have $\left\lceil\frac{n}{3}\right\rceil>\hat{g}(n)$ in those cases.