17th Benelux Mathematical Olympiad

25–27 April 2025 — Liège, Belgium

Solutions

Problem 1. Does there exist a function $f: \mathbb{R} \to \mathbb{R}$ such that

$$f\big(x^2 + f(y)\big) = f(x)^2 - y$$

for all $x, y \in \mathbb{R}$?

Solution. There does not exist such a function. Let us suppose by contradiction it does. By substituting $x \leftarrow 0$, we get

$$f(f(y)) = f(0)^2 - y$$

for all $y \in \mathbb{R}$. Since the right-hand side is bijective, this implies that f is also bijective. Taking $y \leftarrow 0$, we get

$$f\left(x^2 + f(0)\right) = f(x)^2$$

for all $x \in \mathbb{R}$ and so $f(-x)^2 = f(x^2 + f(0)) = f(x)^2$. Since f is injective, we get $f(-x) = -f(x)$ for all $x \neq 0$. Since f is a surjection, there exists $r \in \mathbb{R}$ such that $f(r) = 0$. If $r \neq 0$, $f(-r) = -f(r) = 0 = f(r)$ contradicting the fact that f is injective. So $r = 0$ and $f(0) = 0$. Substituting $(x, y) \leftarrow (1, 0)$ yields $f(1) = f(1)^2$ and so $f(1) = 1$ since $f(0) = 0$ and f is injective. Taking $(x, y) \leftarrow (0, 1)$, we finally get $1 = f(f(1)) = -1$ which is the desired contradiction.

Alternative solution We observe that, for all $y, z \in \mathbb{R}$, we have

$$z \geqslant f(y) \implies f(z) \geq -y.$$

Indeed, we can take $x = \sqrt{z - f(y)}$ and get $f(z) = f(x)^2 - y \geqslant -y$. We deduce that $\lim_{z\to+\infty} f(z) = +\infty$. Indeed, for any $K \in \mathbb{R}$, we have $z \geqslant f(-K) \implies f(z) \geqslant K$.
Let us now fix x, and let $y \to +\infty$. We have $x^2 + f(y) \to +\infty$, hence $f(x^2 + f(y)) \to +\infty$. On the other side, we have $f(x)^2 - y \to -\infty$, a contradiction.

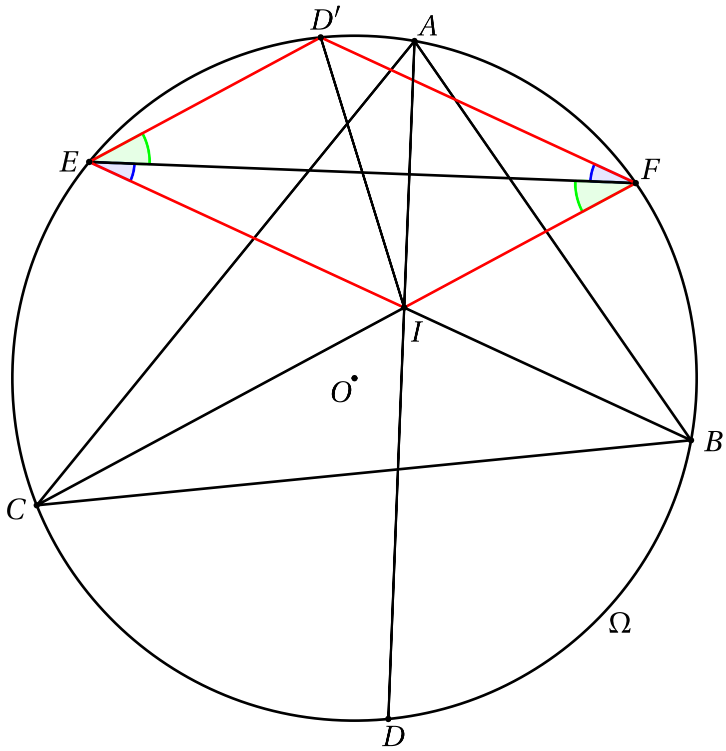
Problem 2. Let $N \geqslant 2$ be a natural number. At a mathematical olympiad training camp, the same N courses are organised every day. Each student takes exactly one of the N courses each day. At the end of the camp, every student has taken each course exactly once, and any two students took the same course on at least one day, but took different courses on at least one other day. What is, in terms of N, the largest possible number of students at the camp?

Solution. The largest number of students at the camp is $(N-1)!$. Since each student takes exactly one course each day and, at the end, has taken each course exactly once, the schedule of a student can be represented by a permutation of the set of the N courses. To show that $(N-1)!$ is possible, we can e.g. assign to each of the $(N-1)!$ students a unique permutation of the $N-1$ first courses and making them all take the N-th course on the last day. It is easy to observe that such a construction satisfies the properties of the statement.

To prove that, for a set S of students, one has $|S| \leqslant (N-1)!$, one first subdivides the set of permutations into disjoint subsets of size N. Two permutations are said to be in the same subset if and only if one can be obtained from the other by cyclically permute the order of the course. Clearly, this is a well-defined subdivision since cyclically permuting twice the order of the courses can be obtained by cyclically permute them only once. Moreover, each of these subsets contains exactly N permutations since there are N cycles of length N. There are thus $\frac{N!}{N} = (N-1)!$ such subsets. If $|S| > (N-1)!$, two students will have their associated permutations in the same subset. However, they cannot be the same permutation (otherwise the two students took the same course every day), and they cannot be obtained by a non-trivial cyclic permutation from each other (otherwise the two students never took the same course).

Problem 3. Let ABC be a triangle with incentre I and circumcircle Ω. Let D, E, F be the midpoints of the arcs $\widehat{BC}, \widehat{CA}, \widehat{AB}$ of Ω not containing A, B, C, respectively. Let D' be the point of Ω diametrically opposite to D. Show that I, D', and the midpoint M of $[EF]$ lie on a line.

Solution.



By definition of D, E and F, we know that ID, IE and IF are the angle bisectors of ABC. Using angles in Ω, we can compute

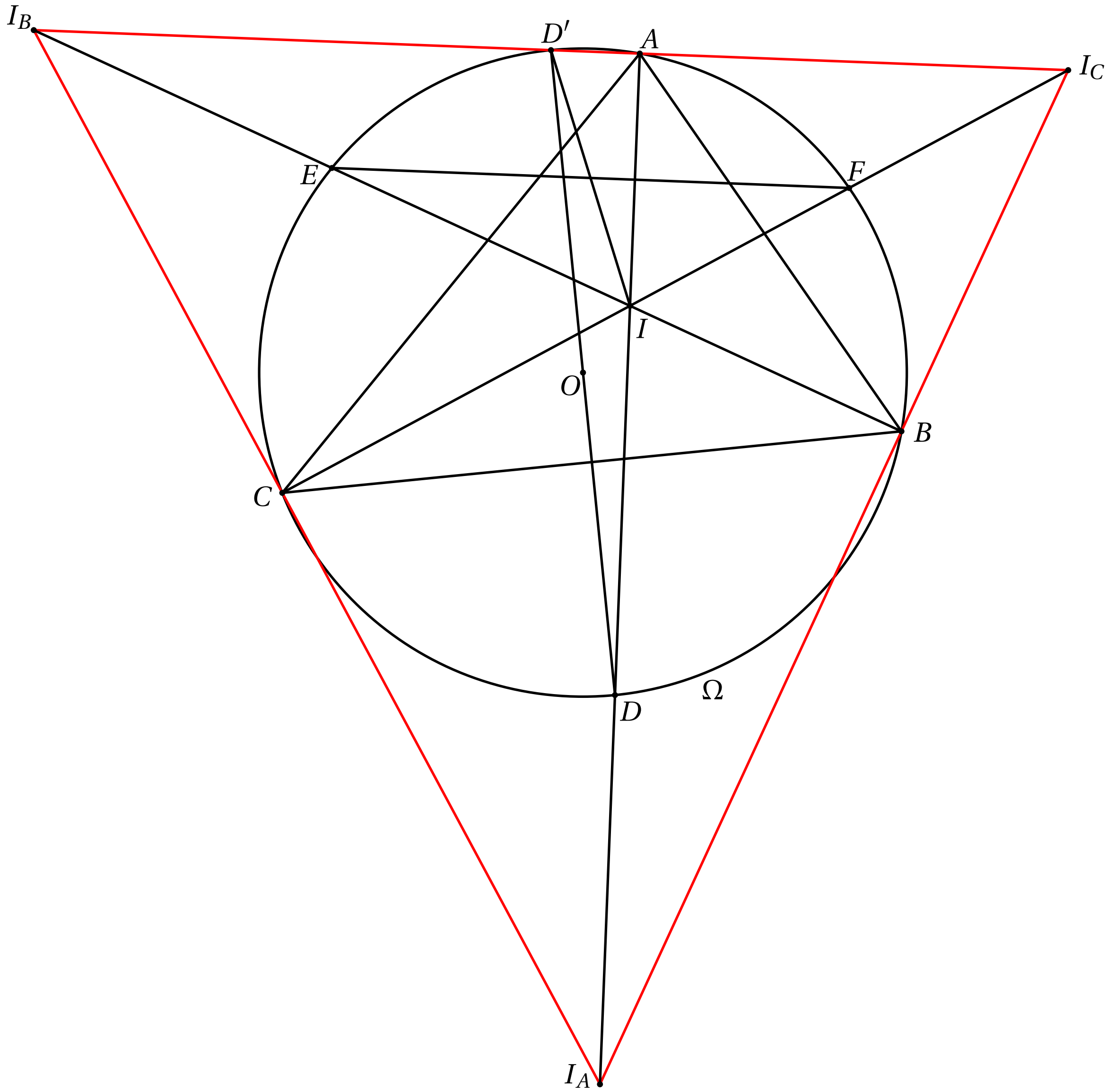
$$\begin{aligned}\widehat{EFI} = \widehat{EFC} = \widehat{EBC} = \frac{\widehat{ABC}}{2} &= 90^\circ - \frac{\widehat{ACB}}{2} - \frac{\widehat{BAC}}{2} \\ &= 90^\circ - \widehat{FCB} - \widehat{BAD} = 90^\circ - \widehat{FEB} - \widehat{BED} = 90^\circ - \widehat{FED} = \widehat{D'EF}\end{aligned}$$

proving that $D'E \parallel FI$. Moreover, we can also compute

$$\begin{aligned}\widehat{FEI} = \widehat{FEB} = \widehat{FCB} = \frac{\widehat{ACB}}{2} &= 90^\circ - \frac{\widehat{ABC}}{2} - \frac{\widehat{CAB}}{2} \\ &= 90^\circ - \widehat{EBC} - \widehat{CAD} = 90^\circ - \widehat{EFC} - \widehat{CFD} = 90^\circ - \widehat{EFD} = \widehat{D'FE}\end{aligned}$$

proving that $EI \parallel D'F$. Therefore, $ED'FI$ is a parallelogram and its diagonals intersect in their midpoints, proving that $D'I$ contains M.

Alternative solution



By definition of D, E and F, we know that ID, IE and IF are the angle bisectors of ABC. If $D' = A$, this means that $\widehat{OAB} = 90° - \widehat{ACB} = \frac{\widehat{CAB}}{2}$ and ABC is isosceles in A. In that case, the symmetry with respect to AI sends E to F and so $M \in AI$. We can thus suppose that $D' \neq A$.
Let us consider I_A, I_B and I_C the excenters of ABC. The lines I_BAI_C, I_CBI_A and I_ACI_B are thus the exterior bisectors of the triangle. Hence, $I_BI_C \perp AI_A$ and, since $\widehat{D'AD} = 90°$, one has $D' \in I_BI_C$. In the triangle $I_AI_BI_C$, the points A, B and C are the feet of the heights and Ω is thus the Euler circle of $I_AI_BI_C$. Since $D' \in [I_BI_C] \cap \Omega$ and $D' \neq A$, it is the midpoint of $[I_BI_C]$. Moreover, E and F, being on the Euler circle and the heights, there are the midpoints of $[I_BI]$ and $[I_CI]$ respectively. The homothety of centre I and ratio $\frac{1}{2}$ sends I_B on E and I_C on F. It thus also sends D' on M, proving that D', M and I are collinear.

Problem 4. Let $a_0, a_1, \ldots, a_{10}$ be integers such that, for each $i \in \{0,1,\ldots,2047\}$, there exists a subset $S \subseteq \{0,1,\ldots,10\}$ with

$$\sum_{j\in S} a_j \equiv i \pmod{2048}.$$

Show that for each $i \in \{0,1,\ldots,10\}$, there is exactly one $j \in \{0,1,\ldots,10\}$ such that a_j is divisible by 2^i but not by 2^{i+1}.

Note: $\sum\limits_{j\in S} a_j$ is the summation notation, for instance, $\sum\limits_{j\in\{2,5\}} a_j = a_2 + a_5$, while, for the empty set $\varnothing$, one defines $\sum\limits_{j\in\varnothing} a_j = 0$.

Solution. We denote by $v_2(a)$ the valuation 2-adic of the integer a. Let us prove by induction the more general statement that, for $n \in \mathbb{N}_{>0}$, if $a_0, a_1, \ldots, a_{n-1}$ are integers such that, for each $i \in \{0,1,\ldots,2^n-1\}$, there exists a subset $S \subseteq \{0,1,\ldots,n-1\}$ with $\sum_{j\in S} a_j \equiv i \pmod{2^n}$, then, for each $i \in \{0,1,\ldots,n-1\}$, there is exactly one $j \in \{0,1,\ldots,n-1\}$ such that $v_2(a_j) = i$. The result then follows by setting $n = 11$. The case $n = 1$ is trivial.
We suppose by induction that the result is true for $n-1$ for prove it for $n \geqslant 2$. Let us first notice that, since there are 2^n elements in $\{0,1,\ldots,2^n-1\}$ and 2^n subsets of $\{0,1,\ldots,n-1\}$, for each $i \in \{0,1,\ldots,2^n-1\}$, there exists exactly one $S_i \subseteq \{0,1,\ldots,n-1\}$ such that $\sum_{j\in S_i} a_j \equiv i \pmod{2^n}$. By summing all these sums for all i, we obtain

$$\sum_{i=0}^{2^n-1} \sum_{j\in S_i} a_j \equiv \sum_{i=0}^{2^n-1} i = \frac{2^n\cdot(2^n-1)}{2} = 2^{n-1}\cdot(2^n-1) \pmod{2^n}.$$

For a fixed $j \in \{0,1,\ldots,n-1\}$, a_j appears in exactly 2^{n-1} of these sums (since each of the $n-1$ remaining indices may or may not be in S_i). Therefore,

$$2^{n-1}\cdot\sum_{j=0}^{n-1} a_j = \sum_{i=0}^{2^n-1}\sum_{j\in S_i} a_j \equiv 2^{n-1}\cdot(2^n-1) \pmod{2^n}$$

which is equivalent to

$$\sum_{j=0}^{n-1} a_j \equiv 2^n - 1 \equiv 1 \pmod{2}.$$

Since $\sum_{j=0}^{n-1} a_j$ is odd, at least one of the a_j's is odd.
Let us suppose by contradiction that at least two of them are odd. We now sum all the $\sum_{j\in S_i} a_j$ for which $\sum_{j\in S_i} a_j$ (or, equivalently, i) is even:

$$\sum_{\substack{i\in\{0,\ldots,2^n-1\}\\ i \text{ is even}}} \sum_{j\in S_i} a_j \equiv \sum_{\substack{i\in\{0,\ldots,2^n-1\}\\ i \text{ is even}}} i = 2^{n-1}\cdot(2^{n-1}-1) \pmod{2^n}$$

For a fixed $j \in \{0,1,\ldots,n-1\}$, a_j appears in exactly 2^{n-2} of these sums. Indeed, since there is at least one $k \in \{0,1,\ldots,n-1\}\setminus\{j\}$ such that a_k is odd, one can consider any $S \subseteq \{0,1,\ldots,n-1\}\setminus\{j,k\}$, add j to it, and potentially also k in order to make the partial sum even (exactly one possibility for each such S). Therefore,

$$2^{n-2}\cdot\sum_{j=0}^{n-1} a_j = \sum_{\substack{i\in\{0,\ldots,2^n-1\}\\ i \text{ is even}}} \sum_{j\in S_i} a_j \equiv 2^{n-1}\cdot(2^{n-1}-1) \pmod{2^n}$$

or equivalently

$$\sum_{j=0}^{n-1} a_j \equiv 2\cdot(2^{n-1}-1) \pmod 4.$$

Hence $\sum_{j=0}^{n-1} a_j$ is even, which is a contradiction. Therefore, there is exactly one $k \in \{0,1,\dots,n-1\}$ such that a_k is odd.
A sum $\sum_{j\in S} a_j$ is odd if and only if $k \in S$. Moreover, the sums $\sum_{j\in S} a_j$ for which $k \notin S$ cover all the even numbers modulo 2^n. Thus, omitting a_k, the $n-1$ integers $\frac{a_0}{2},\dots,\frac{a_{n-1}}{2}$ satisfy the condition of the statement. By the induction hypothesis, for each $i \in \{0,1,\dots,n-2\}$, there is exactly one $j \in \{0,1,\dots,n-1\}\setminus\{k\}$ such that $\nu_2\left(\frac{a_j}{2}\right) = i$, i.e., $\nu_2(a_j) = i+1$. Since $\nu_2(a_k) = 0$, this concludes the proof.
Remark: If one considers the sum of all $\sum_{j\in S_i} a_j$ such that i is odd, one obtains, in the case where at least two a_i's are odd, that $2^{n-2}\cdot\sum_{j=0}^{n-1} a_j \equiv 0 \pmod{2^n}$ also reaching a contradiction.

Alternative solution We do the same induction as in the main Solution. If all a_j are even, then all the sums are even, contradicting the hypothesis. Therefore, without loss of generality, we can assume that a_0 is odd. For each $i \in \{0,1,2,\dots,2^n-1\}$, let $S'_i \subseteq \{0,1,\dots,n-1\}$ be the subset such that $\sum_{j\in S'_i} a_j \equiv i\cdot a_0 \pmod{2^n}$ (this subset exists by the hypothesis and is unique since there are exactly 2^n subsets of $\{0,1,\dots,n-1\}$). By definition, one has $S'_0 = \emptyset$ and $S'_1 = \{0\}$.
Let us prove by induction that, for each $i \in \{0,1,\dots,2^{n-1}-1\}$, one has $0 \notin S'_{2i}$ but $0 \in S'_{2i+1}$. The case $i = 0$ is trivial. For $i > 0$, if we suppose that $0 \in S'_{2i-1}$, we prove that $0 \notin S'_{2i}$. Indeed, if $0 \in S'_{2i}$, then $\sum_{j\in S'_{2i}\setminus\{0\}} a_j \equiv 2ia_0 - a_0 = (2i-1)a_0 \equiv \sum_{j\in S'_{2i-1}} a_j \pmod{2^n}$ and so $S'_{2i}\setminus\{0\} = S'_{2i-1}$ by uniqueness. But $0 \in S'_{2i-1}$ so this is a contradiction, proving that $0 \notin S'_{2i}$. Then, $\sum_{j\in S'_{2i}\cup\{0\}} a_j \equiv 2ia_0 + a_0 = (2i+1)a_0 \equiv \sum_{j\in S'_{2i+1}} a_j \pmod{2^n}$. By uniqueness, $S'_{2i}\cup\{0\} = S'_{2i+1}$ and so $0 \in S'_{2i+1}$.
Since a_0 is odd, it is invertible modulo 2^n and so 0, a_0, $2a_0$,…, $(2^n-1)a_0$ is a permutation of 0, 1, 2,…, 2^n-1 modulo 2^n. This shows that the subsets S'_0, S'_2, S'_4,…, S'_{2^n-2} are all distinct. There are thus 2^{n-1} subsets S'_{2i} and they are a precisely the subsets of $\{1,2,\dots,n-1\}$ (since there are also 2^{n-1} such subsets). Moreover, their corresponding sums are all even. Since this list of subsets contains $\{1\}$, $\{2\}$,…, $\{n-1\}$, this shows that $a_1, a_2, \dots, a_{n-1}$ are all even.
We then conclude the induction as in the main Solution.

Alternative solution We do the same induction as in the main Solution (except that we prove that for each $i \in \{0,1,\dots,n-1\}$, there exists $j \in \{0,1,\dots,n-1\}$ with $\nu_2(a_j) = i$, uniqueness follows immediately). For the induction step, let us consider the polynomial

$$P(X) = \prod_{j=0}^{n-1}\left(X^{a_j}+1\right).$$

The condition in the statement implies that

$$P(X) \equiv \sum_{i=0}^{2^n-1} X^i \pmod{X^{2^n}-1}.$$

Since

$$\sum_{i=0}^{2^n-1} X^i = \frac{X^{2^n}-1}{X-1} = \frac{X^{2^{n-1}}-1}{X-1}\cdot\left(X^{2^{n-1}}+1\right),$$

we know that $P(X)$ is divisible by $X^{2^{n-1}}+1$. Let $\omega \in \mathbb{C}$ be a primitive 2^n-th root of unity. Thus $\omega^{2^{n-1}} = -1$ and $P(\omega) = 0$. There exists thus $j \in \{0,1,\dots,n-1\}$ such that $\omega^{a_j} = -1$. This means that $a_j = (2k+1)2^{n-1}$ for some integer k, i.e., $\nu_2(a_j) = n-1$. It is then obvious that the other ones satisfy the property for $n-1$ and we conclude by the inductive hypothesis.